

# MULTIFRACTAL ANALYSIS OF THE DIVERGENCE OF FOURIER SERIES: THE EXTREME CASES

FRÉDÉRIC BAYART, YANICK HEURTEAUX

**ABSTRACT.** We study the size, in terms of the Hausdorff dimension, of the subsets of  $\mathbb{T}$  such that the Fourier series of a generic function in  $L^1(\mathbb{T})$ ,  $L^p(\mathbb{T})$  or in  $\mathcal{C}(\mathbb{T})$  may behave badly. Genericity is related to the Baire category theorem or to the notion of prevalence. This paper is a continuation of [2].

## 1. INTRODUCTION

This paper, which can be seen as a continuation of [2], deals with the divergence of Fourier series of functions in  $L^p(\mathbb{T})$ ,  $p \geq 1$ , where  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ , or in  $\mathcal{C}(\mathbb{T})$ , from the multifractal point of view. More precisely, let  $f$  be in  $L^p(\mathbb{T})$ , or in  $\mathcal{C}(\mathbb{T})$ , and let  $(S_n f)_{n \geq 0}$  the sequence of partial sums of its Fourier series. We are interested in the size of the sets of the real numbers  $x$  such that  $(S_n f(x))_{n \geq 0}$  diverges with a prescribed growth.

We will measure the size of subsets of  $\mathbb{T}$  using the Hausdorff dimension. Let us recall the relevant definitions (we refer to [5] and to [8] for more on this subject). If  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing continuous function satisfying  $\phi(0) = 0$  ( $\phi$  is called a *dimension function* or a *gauge function*), the  $\phi$ -Hausdorff outer measure of a set  $E \subset \mathbb{R}^d$  is

$$\mathcal{H}^\phi(E) = \lim_{\varepsilon \rightarrow 0} \inf_{r \in R_\varepsilon(E)} \sum_{B \in r} \phi(|B|),$$

where  $R_\varepsilon(E)$  is the set of (countable) coverings of  $E$  with balls  $B$  of diameter  $|B| \leq \varepsilon$ . When  $\phi_s(x) = x^s$ , we write for short  $\mathcal{H}^s$  instead of  $\mathcal{H}^{\phi_s}$ . The Hausdorff dimension of a set  $E$  is defined by

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0; \mathcal{H}^s(E) > 0\} = \inf\{s > 0; \mathcal{H}^s(E) = 0\}.$$

The first result studying the Hausdorff dimension of the divergence sets of Fourier series is due to J-M. Aubry [1].

**Theorem 1.1.** *Let  $f \in L^p(\mathbb{T})$ ,  $1 < p < +\infty$ . If  $\beta \geq 0$ , define*

$$\mathcal{E}(\beta, f) = \left\{ x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

*Then  $\dim_{\mathcal{H}}(\mathcal{E}(\beta, f)) \leq 1 - \beta p$ . Conversely, given a set  $E$  such that  $\dim_{\mathcal{H}}(E) < 1 - \beta p$ , there exists a function  $f \in L^p(\mathbb{T})$  such that, for any  $x \in E$ ,  $\limsup_{n \rightarrow +\infty} n^{-\beta} |S_n f(x)| = +\infty$ .*

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This result motivated us to introduce in [2] the notion of divergence index. For a given function  $f \in L^p(\mathbb{T})$  and a given point  $x_0 \in \mathbb{T}$ , we can define  $\beta(x_0)$  as the infimum of the nonnegative real numbers  $\beta$  such that  $|S_n f(x_0)| = O(n^\beta)$ . The real number  $\beta(x_0)$  will be called the *divergence index* of the Fourier series of  $f$  at point  $x_0$ . It is well-known that, for any function  $f \in L^p(\mathbb{T})$  ( $1 \leq p < +\infty$ ) and any point  $x_0 \in \mathbb{T}$ ,  $0 \leq \beta(x_0) \leq 1/p$  (see [11]). Moreover, when  $p > 1$ , Carleson's theorem implies that  $\beta(x_0) = 0$  almost surely. In [2], we gave precise estimates on the size of the level sets of the function  $\beta$ . These are defined as

$$\begin{aligned} E(\beta, f) &= \{x \in \mathbb{T}; \beta(x) = \beta\} \\ &= \left\{x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log n} = \beta\right\}. \end{aligned}$$

**Theorem 1.2** ([2]). *Let  $1 < p < +\infty$ . For quasi-all functions  $f \in L^p(\mathbb{T})$ , for any  $\beta \in [0, 1/p]$ ,  $\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p$ .*

The terminology "quasi-all" used here is relative to the Baire category theorem. It means that this property is true for a residual set of functions in  $L^p(\mathbb{T})$ .

In the case of continuous functions, the situation breaks down dramatically. If  $(D_n)_{n \geq 0}$  denotes the Dirichlet kernel, we can first observe that, when  $f \in \mathcal{C}(\mathbb{T})$ ,

$$\|S_n f\|_{\infty} \leq \|D_n\|_1 \|f\|_{\infty} \leq C \|f\|_{\infty} \log n.$$

This motivated us in [2] to introduce the following level sets:

$$\begin{aligned} \mathcal{F}(\beta, f) &= \left\{x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} (\log n)^{-\beta} |S_n f(x)| > 0\right\} \\ F(\beta, f) &= \left\{x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log \log n} = \beta\right\}. \end{aligned}$$

Whereas, on  $L^p(\mathbb{T})$ ,  $1 < p < +\infty$ , the divergence index takes its biggest value ( $\beta(x) = 1/p$ ) on small sets, this is far from being the case on  $\mathcal{C}(\mathbb{T})$ , as the following very surprising result indicates.

**Theorem 1.3** ([2]). *For quasi-all functions  $f \in \mathcal{C}(\mathbb{T})$ , for any  $\beta \in [0, 1]$ ,  $F(\beta, f)$  is non-empty and has Hausdorff dimension 1.*

However, several questions were left open in [2].

**Question 1: what happens on  $L^1(\mathbb{T})$ ?** In view of the differences between  $L^p(\mathbb{T})$ ,  $p \in (1, +\infty)$ , and  $\mathcal{C}(\mathbb{T})$ , it seems *a priori* not clear what situation should be expected on  $L^1(\mathbb{T})$ . Moreover, Carleson's theorem is false on  $L^1(\mathbb{T})$  and Kolmogorov Theorem ensures that there exist functions in  $L^1(\mathbb{T})$  with everywhere divergent Fourier series.

The proof of Theorem 1.2 proceeds in two steps. In a first time, we build a residual set of functions in  $L^p(\mathbb{T})$  such that, if  $f$  lies in this residual set and if  $0 \leq \beta \leq 1/p$ ,  $\dim_{\mathcal{H}}(E(\beta, f)) \geq 1 - \beta p$ . In a second time, we use Theorem 1.1 to conclude that necessarily  $\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p$ . The first step works as well in  $L^1(\mathbb{T})$  and the trouble comes from Aubry's result, which uses the Carleson Hunt maximal inequality. In Section 2, we

succeed to overcome this difficulty by proving a (very weak!) version of Carleson's maximal inequality in  $L^1(\mathbb{T})$  which is sufficient to prove the analogue of Theorem 1.1. Thus, we will show that

**Theorem 1.4.** *For quasi-all functions  $f \in L^1(\mathbb{T})$ , for any  $\beta \in [0, 1]$ ,*

$$\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta.$$

**Question 2: what about the size of the set of multifractal functions?** Theorem 1.2 and Theorem 1.4 say that, in  $L^p(\mathbb{T})$  ( $p \geq 1$ ), the set of multifractal functions is big in a topological sense. One can ask if it remains big for other points of view. We deal here with an infinite-dimensional version of the notion of "almost-everywhere". This notion, called *prevalence*, has been introduced by J. Christensen in [4] and has been widely studied since then. In multifractal analysis, some properties which are true on a dense  $G_\delta$ -set are also prevalent (see for instance [7] or [6]), whereas some are not (see for instance [7] or [10]). This motivated us to examine Theorem 1.2 and Theorem 1.4 under this point of view.

**Definition 1.5.** Let  $E$  be a complete metric vector space. A Borel set  $A \subset E$  is called *Haar-null* if there exists a compactly supported probability measure  $\mu$  such that, for any  $x \in E$ ,  $\mu(x + A) = 0$ . If this property holds, the measure  $\mu$  is said to be *transverse* to  $A$ . A subset of  $E$  is called *Haar-null* if it is contained in a Haar-null Borel set. The complement of a Haar-null set is called a *prevalent* set.

The following results enumerate important properties of prevalence and show that this notion supplies a natural generalization of "almost every" in infinite-dimensional spaces:

- If  $A$  is Haar-null, then  $x + A$  is Haar-null for every  $x \in E$ .
- If  $\dim(E) < +\infty$ ,  $A$  is Haar-null if and only if it is negligible with respect to the Lebesgue measure.
- Prevalent sets are dense.
- The intersection of a countable collection of prevalent sets is prevalent.
- If  $\dim(E) = +\infty$ , compact subsets of  $E$  are Haar-null.

In Section 3, we will prove the following result.

**Theorem 1.6.** *Let  $1 \leq p < +\infty$ . The set of functions  $f \in L^p(\mathbb{T})$  such that, for any  $\beta \in [0, 1/p]$ ,  $\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p$ , is prevalent.*

Thus, almost every function in  $L^p(\mathbb{T})$  is multifractal with respect to the summation of its Fourier series.

**Question 3: can we say more on  $\mathcal{C}(\mathbb{T})$ ?** Theorem 1.3 implies that there exists a residual subset  $A \subset \mathcal{C}(\mathbb{T})$  such that, if  $f \in A$  and if  $\beta < 1$ , one can find a set  $E \subset \mathbb{T}$  with Hausdorff dimension 1 such that

$$(1) \quad \limsup_{n \rightarrow +\infty} \frac{|S_n f(x)|}{(\log n)^\beta} = +\infty \text{ for any } x \in E.$$

On the other hand, we know that, for any fixed  $f \in \mathcal{C}(\mathbb{T})$ ,  $\|S_n f\|_\infty$  is negligible compared to  $\log n$  and that, conversely, given any sequence  $(\delta_n)_{n \geq 2}$  of positive real numbers going

to zero, we can find  $f \in \mathcal{C}(\mathbb{T})$  such that

$$\limsup_{n \rightarrow +\infty} \frac{|S_n f(0)|}{\delta_n \log n} = +\infty.$$

These statements can be found for example in [11]. It seems then natural to ask whereas this property can be ensured in a set with Hausdorff dimension equal to 1 ( (1) meaning that this is true when  $\delta_n = (\log n)^{\beta-1}$ ,  $0 < \beta < 1$ ). This is indeed true.

**Theorem 1.7.** *Let  $(\delta_n)_{n \geq 2}$  be a sequence of positive real numbers going to zero. For quasi-all functions  $f \in \mathcal{C}(\mathbb{T})$ , there exists  $E \subset \mathbb{T}$  with Hausdorff dimension 1 such that, for any  $x \in E$ ,*

$$\limsup_{n \rightarrow +\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty.$$

The same result also holds in a prevalent subset of  $\mathcal{C}(\mathbb{T})$ .

**Theorem 1.8.** *Let  $(\delta_n)_{n \geq 2}$  be a sequence of positive real numbers going to zero. For almost every function  $f \in \mathcal{C}(\mathbb{T})$ , there exists  $E \subset \mathbb{T}$  with Hausdorff dimension 1 such that, for any  $x \in E$ ,*

$$\limsup_{n \rightarrow +\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty.$$

The proof of Theorems 1.7 and 1.8 are proposed in Section 4.

## 2. MULTIFRACTAL ANALYSIS OF THE DIVERGENCE OF FOURIER SERIES IN $L^1(\mathbb{T})$

We first recall some basic facts on Fourier series and Fourier transforms in  $L^p$ . Let  $\xi \in \mathbb{R}$  and  $e_\xi : t \mapsto e^{2\pi i \xi t}$ . The Fourier transform of  $f \in L^1(\mathbb{R})$  is the continuous function

$$\hat{f} : \xi \mapsto \int_{\mathbb{R}} f(x) \bar{e}_\xi(x) dx.$$

The operator makes also sense in the space  $L^p(\mathbb{R})$  when  $1 \leq p < +\infty$ . In that case,  $\hat{f} \in L^q(\mathbb{R})$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . In  $L^p(\mathbb{R})$  we can define the band-limiting operator  $S_n$  by

$$\widehat{S_n f} = \mathbf{1}_{[-n, n]} \hat{f}.$$

It is well known that, on  $L^p(\mathbb{R})$ , the projections  $(S_n)_{n \geq 0}$  are uniformly bounded; this is the Riesz theorem. This is not the case on  $L^1(\mathbb{R})$ . However, there exists some absolute constant  $C > 0$  such that, for any  $n \geq 2$  and any  $f \in L^1(\mathbb{R})$ ,

$$\|S_n f\|_1 \leq C \log n \|f\|_1.$$

A function  $g \in L^1(\mathbb{T})$  is identified to a 1-periodic function on  $\mathbb{R}$ . Its Fourier transform is the tempered distribution

$$\hat{g} = \sum_{k \in \mathbb{Z}} \langle g, e_k \rangle \delta_k,$$

where  $\langle g, e_k \rangle = \int_{\mathbb{T}} g(t) \bar{e}_k(t) dt$  are the Fourier coefficients of  $g$  and  $\delta_k$  denotes the Dirac mass at point  $k$ . If  $g \in L^1(\mathbb{T})$ , the band limiting operator corresponds to taking the partial sum of the Fourier series,

$$S_n g : t \mapsto \sum_{k=-n}^n \langle g, e_k \rangle e_k(t).$$

We can also write  $S_n g = D_n * g$  where

$$D_n(t) = \sum_{k=-n}^n e_k(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}$$

is the Dirichlet kernel and the Riesz theorem always occurs in this context.

Let us also recall the definition of  $\sigma_n g$ , the  $n$ -th Féjer sum of  $g$ , namely

$$\sigma_n(g) = \frac{1}{n}(S_0 g + \cdots + S_{n-1} g).$$

We write  $\mathcal{E}_n(\mathbb{T}) := S_n(L^1(\mathbb{T}))$  the set of trigonometric polynomials of degree less than  $n$  and  $\mathcal{E}_n(\mathbb{R}) := S_n(L^1(\mathbb{R}))$ . The classical Nikolsky inequality (see for example [9]) says that if  $P \in \mathcal{E}_n(\mathbb{T})$  or  $P \in \mathcal{E}_n(\mathbb{R})$  and  $1 \leq p \leq q \leq \infty$ , then

$$\|P\|_q \leq n^{\frac{1}{p} - \frac{1}{q}} \|P\|_p.$$

Our first lemma will be helpful to control a function which is locally a Dirichlet kernel.

**Lemma 2.1.** *There exists a constant  $A > 0$  such that, for any  $N \geq 2$ , for any measurable function  $n : \mathbb{T} \rightarrow \{1, \dots, N\}$ , for any  $t \in \mathbb{T}$ , then*

$$\int_{\mathbb{T}} |D_{n(x)}(x - t)| dx \leq A \log N.$$

*Proof.* It is obvious from the above expression of  $D_n$  that, if  $k \leq N$  and if  $u \in [-1/2, 1/2]$ ,

$$|D_k(u)| \leq \begin{cases} CN \\ \frac{C}{|u|} \end{cases}$$

for some absolute constant  $C > 0$ . We then split the integral into two parts:

$$\int_{|x-t| \leq 1/N} |D_{n(x)}(x - t)| dx \leq 2CN \frac{1}{N}$$

and

$$\int_{1/N < |x-t| \leq 1/2} |D_{n(x)}(x - t)| dx \leq C \int_{1/N < |x-t| \leq 1/2} \frac{dx}{|x - t|} \leq 2C \log N.$$

□

Writing  $S_{n(x)} f(x) = (f \star D_{n(x)})(x)$  and using Fubini's theorem, it is straightforward to deduce the following inequality on partial sums of Fourier series of  $L^1$ -functions.

**Lemma 2.2.** *There exists a constant  $A > 0$  such that, for any  $N \geq 2$ , for any measurable function  $n : \mathbb{T} \rightarrow \{1, \dots, N\}$ , for any  $f \in L^1(\mathbb{T})$ , then*

$$\int_{\mathbb{T}} |S_{n(x)} f(x)| dx \leq A \log N \|f\|_1.$$

We are now ready to prove the following weak version of the maximal inequality of Carleson and Hunt, on  $L^1(\mathbb{T})$ .

**Corollary 2.3.** *Let  $\alpha > 0$ . There exists  $C := C_\alpha > 0$  such that, for any  $f \in L^1(\mathbb{T})$ ,*

$$\int_{\mathbb{T}} \sup_{n \geq 2} \frac{|S_n f(x)|}{(\log n)^{1+\alpha}} dx \leq C \|f\|_1.$$

*Proof.* Using the monotone convergence theorem, we first observe that it is sufficient to prove that, for any  $N \geq 2$ ,

$$(2) \quad \int_{\mathbb{T}} \sup_{2 \leq n \leq N} \frac{|S_n f(x)|}{(\log n)^{1+\alpha}} dx \leq C \|f\|_1$$

where, of course,  $C$  does not depend on  $N$ . Now, we take a measurable function  $n : \mathbb{T} \rightarrow \mathbb{N} \setminus \{0, 1\}$  not necessarily bounded, and observe that (2) will be proved if we are able to show that

$$\int_{\mathbb{T}} \frac{|S_{n(x)} f(x)|}{(\log n(x))^{1+\alpha}} dx \leq C \|f\|_1$$

for some constant  $C$  independent of the function  $n$ . If  $k \geq 0$ , let

$$A_k = \{x \in \mathbb{T}; 2^{2^k} \leq n(x) < 2^{2^{k+1}}\}.$$

Lemma 2.2 ensures that

$$\begin{aligned} \int_{\mathbb{T}} \frac{|S_{n(x)} f(x)|}{(\log n(x))^{1+\alpha}} dx &= \sum_{k \geq 0} \int_{A_k} \frac{|S_{n(x)} f(x)|}{(\log n(x))^{1+\alpha}} dx \\ &\leq \sum_{k \geq 0} \frac{1}{(2^k \log 2)^{1+\alpha}} \int_{A_k} |S_{n(x)} f(x)| dx \\ &\leq \sum_{k \geq 0} C \frac{2^{k+1} \log 2}{2^{k(1+\alpha)} (\log 2)^{1+\alpha}} \|f\|_1 \\ &= C_\alpha \|f\|_1. \end{aligned}$$

□

The following lemma is inspired by Aubry's paper. It means that, as soon as a trigonometric polynomial is large at some point  $a \in \mathbb{T}$ , it is also large in small intervals around  $a$ , with a rather good control of the  $L^p$ -norm.

**Lemma 2.4.** *Let  $p \geq 1$  and  $\varepsilon > 0$ . There exists  $\delta > 0$  such that, if  $n$  is large enough, if  $P \in \mathcal{E}_n(\mathbb{T})$  and if  $a \in \mathbb{T}$  is such that  $|P(a)| \geq \|P\|_p$ , then, for any interval  $I$  with center  $a$  and with length  $|I| \leq \frac{1}{n}$ ,*

$$\|P\|_{L^p(I)} \geq \delta |P(a)| \times |I|^{1/p} \times \begin{cases} \frac{1}{(\log n)^{(1+\varepsilon)/p}} & \text{provided } p > 1 \\ \frac{1}{(\log n)^{1+\varepsilon} \log(1/|I|)} & \text{provided } p = 1. \end{cases}$$

*Remarks:*

- Such a point  $a$  does exist because  $P$  is continuous.
- In fact, we will only need the lemma in the case  $p = 1$ , but we give the general case for completeness.

*Proof of Lemma 2.4.* Without loss of generality, we may assume that  $a = 0$ . The idea is to localize  $P$  around 0, and to use Nikolsky inequality to estimate the  $L^p$ -norm knowing the  $L^\infty$ -norm. Let  $\gamma \in (0, 1)$  such that  $\gamma(1 + \varepsilon) > 1$ . We introduce a function  $w$  with

support in  $[-1, 1]$  satisfying  $0 \leq w \leq 1$ ,  $w(0) = 1$  and for which there exist two strictly positive constants  $D$  and  $E$  such that

$$\forall \xi \in \mathbb{R}, \quad |\hat{w}(\xi)| \leq D e^{-E|\xi|^\gamma}.$$

It is a classical result in Fourier analysis that such a function does exist (see e.g. [1, Lemma 6]). We then set  $w_I(x) = w(x/|I|)$ . We decompose  $Pw_I$  as  $f_1 + f_2$  with  $f_1 = S_N Pw_I$  and  $N = \lfloor |I|^{-1}(\log n)^{1+\varepsilon} \rfloor$ , the integer part of  $|I|^{-1}(\log n)^{1+\varepsilon}$ . On the one hand, if  $p > 1$  we get

$$\begin{aligned} \|f_1\|_\infty &\leq N^{1/p} \|f_1\|_p \text{ (Nikolsky inequality)} \\ &\leq C_p |I|^{-1/p} (\log n)^{(1+\varepsilon)/p} \|Pw_I\|_p \text{ (Riesz theorem)} \\ &\leq C_p |I|^{-1/p} (\log n)^{(1+\varepsilon)/p} \|P\|_{L^p(I)}. \end{aligned}$$

When  $p = 1$ , we have to add the norm of the Riesz projection, and we get

$$\|f_1\|_\infty \leq C_1 |I|^{-1} (\log n)^{1+\varepsilon} \log(1/|I|) \|P\|_{L^1(I)}.$$

On the other hand, we may write

$$\begin{aligned} \hat{f}_2(\xi) &= \mathbf{1}_{\{|\xi| > N\}}(\xi) (\hat{P} \star \hat{w}_I)(\xi) \\ &= \sum_{j=-n}^n \mathbf{1}_{\{|\xi| > N\}}(\xi) \hat{P}(j) \hat{w}_I(\xi - j). \end{aligned}$$

Now, if  $n$  is large enough and  $j \leq n$ , we have

$$\begin{aligned} \int_{|\xi| > N} |\hat{w}_I(\xi - j)| d\xi &\leq \int_{|\xi| > \frac{1}{2}|I|^{-1}(\log n)^{1+\varepsilon}} |\hat{w}_I(\xi)| d\xi \\ &= \int_{|\xi| > \frac{1}{2}(\log n)^{1+\varepsilon}} |\hat{w}(\xi)| d\xi. \end{aligned}$$

Observe that

$$\int_A^{+\infty} e^{-E\xi^\gamma} d\xi = \frac{1}{\gamma} \int_{A^\gamma}^{+\infty} e^{-Et} t^{1/\gamma-1} dt \leq C e^{-(E/2)A^\gamma}.$$

It follows easily that

$$\int_{|\xi| > N} |\hat{w}_I(\xi - j)| d\xi \leq C n^{-2}$$

provided  $n$  is large enough. This implies

$$\begin{aligned} \|f_2\|_\infty \leq \|\hat{f}_2\|_1 &\leq C n^{-2} \sum_{j=-n}^n |\hat{P}(j)| \\ &\leq C n^{-2} (2n+1) \|P\|_1 \\ &\leq C n^{-2} (2n+1) \|P\|_p \\ &\leq \frac{1}{2} \|P\|_p \end{aligned}$$

provided  $n$  is large enough. If we recall that  $|P(0)| \geq \|P\|_p$ , we get

$$\|f_1\|_\infty \geq |P(0)| - \|f_2\|_\infty \geq \frac{1}{2} |P(0)|$$

and the result follows from the above estimates of  $\|f_1\|_\infty$ .  $\square$

We can now conclude by proving the following proposition (Proposition 2.5) and its corollary on the Hausdorff dimension of  $E(\beta, f)$  (Corollary 2.6). Recall that it is all that we need to obtain Theorem 1.4 since the construction done in [2] is always true when  $p = 1$  and shows that there exists a residual set of functions  $f \in L^1(\mathbb{T})$  with  $\dim_{\mathcal{H}}(E(\beta, f)) \geq 1 - \beta$  for any  $\beta \in [0, 1]$ .

**Proposition 2.5.** *Let  $f \in L^1(\mathbb{T})$  and  $\tau : (0, +\infty) \rightarrow (0, +\infty)$  be an increasing function. Define*

$$E(\tau, f) := \left\{ x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} \frac{|S_n f(x)|}{\tau(n)} = +\infty \right\}.$$

*If  $\nu > 3$  and if  $\phi$  is a dimension function satisfying  $c_1 s \leq \phi(s) \leq c_2 \frac{s\tau(s^{-1})}{\log(s^{-1})^\nu}$ , then*

$$\mathcal{H}^\phi(E(\tau, f)) = 0.$$

*Proof.* Let  $M > 0$  and  $\varepsilon = \nu - 3$ . Define

$$E_M(\tau, f) = \left\{ x \in \mathbb{T}; \limsup_{n \rightarrow +\infty} \frac{|S_n f(x)|}{\tau(n)} > M \right\}.$$

If  $x \in E_M(\tau, f)$ , one can find  $n_x$  as large as we want such that  $|S_{n_x} f(x)| \geq M\tau(n_x)$ . Set  $I_x = \left[ x - \frac{1}{2n_x}, x + \frac{1}{2n_x} \right]$  and observe that  $\|S_{n_x} f\|_1 \leq C(\log n_x)$ . The hypothesis on the function  $\tau$  implies that, if  $n_x$  is large enough,  $\|S_{n_x} f\|_1 \leq |S_{n_x} f(x)|$ . We can then apply Lemma 2.4 and we get

$$\|S_{n_x} f\|_{L^1(I_x)} \geq \delta \frac{M\tau(n_x)}{n_x(\log n_x)^{2+\varepsilon/2}}.$$

$(I_x)_{x \in E_M(\tau, f)}$  is a covering of  $E_M(\tau, f)$ . We can extract a Vitali's covering, namely a countable family of disjoint intervals  $(I_i)_{i \in \mathbb{N}}$ , of length  $1/n_i$ , such that  $E_M(\tau, f) \subset \bigcup_{i \in \mathbb{N}} 5B_i$ . Then, Corollary 2.3 implies

$$\begin{aligned} C\|f\|_1 &\geq \int_{\mathbb{T}} \sup_{n \geq 2} \frac{|S_n f(x)|}{(\log n)^{1+\varepsilon/2}} dx \\ &\geq \sum_i \int_{I_i} \frac{|S_{n_i} f(x)|}{(\log n_i)^{1+\varepsilon/2}} dx \\ &\geq \delta M \sum_i \frac{|I_i| \tau(1/|I_i|)}{(\log(1/|I_i|))^{3+\varepsilon}}. \end{aligned}$$

This yields  $\sum_i \phi(5|I_i|) \leq \frac{C\|f\|_1}{\delta M}$  (we recall that  $\tau$  is increasing), with  $C$  another absolute constant and  $M > 0$  as large as we want. Hence,  $\mathcal{H}^\phi(E_M(\tau, f)) \leq \frac{C\|f\|_1}{\delta M}$  (the length of the intervals of the covering can be arbitrarily small). This in turn implies  $\mathcal{H}^\phi(E(\tau, f)) = 0$ , since  $E(\tau, f) = \bigcap_{M>0} E_M(\tau, f)$ .  $\square$

By applying the previous proposition to  $\tau(s) = s^\beta$  and  $\phi(s) = s^{1-\beta}/\log(s^{-1})^4$ , we get:

**Corollary 2.6.** *For any  $f \in L^1(\mathbb{T})$  and any  $\beta \in [0, 1]$ ,  $\dim_{\mathcal{H}}(E(\beta, f)) \leq 1 - \beta$ .*



## 3. PREVALENCE OF MULTIFRACTAL BEHAVIOUR

**3.1. Strategy.** In all this part,  $p$  is a fixed real number such that  $1 \leq p < +\infty$ . To prove that a set  $A \subset E$  is Haar-null, the Lebesgue measure on the unit ball of a finite-dimensional subspace  $V$  can often play the role of the transverse measure. Precisely, if there exists a finite-dimensional subspace  $V$  of  $E$  such that, for any  $x \in E$ ,  $V \cap (x + A)$  has full Lebesgue-measure, then  $A$  is prevalent. Such a finite-dimensional subspace  $V$  is called a *probe* for  $A$ . Of course, it is the same to prove that for any  $x \in E$ ,  $(x + V) \cap A$  has full Lebesgue-measure.

We shall use this property to prove prevalence. More precisely, we shall first prove that, for a fixed  $\beta \in [0, 1/p]$ , the set of functions  $f$  in  $L^p(\mathbb{T})$  satisfying  $\dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p$  is prevalent. Then we will conclude because a countable intersection of prevalent sets is prevalent.

**3.2. The construction of saturating functions with disjoint spectra.** In this subsection,  $\alpha > 1$  is fixed. For  $j \geq 1$ , we define  $J = [j/\alpha] + 1$ , which is smaller than  $j - 2$  if  $j$  is large enough, say  $j \geq j_\alpha$ . For  $0 \leq K \leq 2^J - 1$ , we define the dyadic intervals

$$I_{K,j} := \left[ \frac{K}{2^J} - \frac{1}{2^j}, \frac{K}{2^J} + \frac{1}{2^j} \right].$$

We also define

$$\mathbf{I}_j := \bigcup_{K=0}^{2^J-1} I_{K,j} \quad \text{and} \quad \mathbf{I}'_j := \bigcup_{K=0}^{2^J-1} 2I_{K,j}.$$

The condition  $j \geq j_\alpha$  ensures that the  $2I_{K,j}$  do not overlap. We finally introduce  $D_\alpha$  the set of real numbers in  $[0, 1]$  which are  $\alpha$ -approximable by dyadics. Namely,  $x \in [0, 1]$  belongs to  $D_\alpha$  if there exist two sequences of integers  $(k_n)_{n \geq 0}$  and  $(j_n)_{n \geq 0}$  such that

$$\left| x - \frac{k_n}{2^{j_n}} \right| \leq \frac{1}{2^{\alpha j_n}}.$$

It is easy to check that  $D_\alpha$  is contained in  $\limsup_{j \rightarrow +\infty} \mathbf{I}_j$ . Indeed, let  $x \in D_\alpha$ . One may find  $J$  as large as we want and  $K$  such that  $|x - K/2^J| \leq 1/2^{\alpha J}$ . Let  $j$  be an integer such that  $J - 1 = [j/\alpha]$  (such an integer exists because  $\alpha \geq 1$ ). We get

$$\left| x - \frac{K}{2^J} \right| \leq \frac{1}{2^j}.$$

Finally,  $x \in \mathbf{I}_j$ . Furthermore, it is well-known that  $\dim_{\mathcal{H}}(D_\alpha) = 1/\alpha$  and even that  $\mathcal{H}^{1/\alpha}(D_\alpha) = +\infty$  (see for instance [3] and the mass transference principle). It follows that

$$\dim_{\mathcal{H}} \left( \limsup_{j \rightarrow +\infty} \mathbf{I}_j \right) \geq \frac{1}{\alpha}.$$

We are going to build finite families of functions which behave badly on each  $\mathbf{I}_j$ , and which have disjoint spectra. The starting point is a modification of the basic construction of [2].

**Lemma 3.1.** *Let  $j \geq j_\alpha$  and  $J = [j/\alpha] + 1$ . There exists a trigonometric polynomial  $P_j$  with spectrum contained in  $(0, 2^{j+1} - 1]$  such that*

$$\bullet \quad \|P_j\|_p \leq 1$$

- $|P_j(x)| \geq C2^{-(J-j)/p}$  for any  $x \in \mathbf{I}_j$

where the constant  $C$  is independant of  $j$ .

*Proof.* Let  $\chi_j$  be a continuous piecewise linear function equal to 1 on  $\mathbf{I}_j$ , equal to 0 outside  $\mathbf{I}'_j$  and satisfying  $0 \leq \chi_j \leq 1$  and  $\|\chi'_j\|_\infty \leq 2^j$ .  $P_j$  is defined by

$$P_j := 2^{-(J-j+2)/p} e_{2^j} \sigma_{2^j} \chi_j.$$

The  $L^p$ -norm of  $P_j$  is clearly less than or equal to 1 (observe that the measure of  $\mathbf{I}'_j$  is  $2^{J-j+2}$ ). Applying Lemma 1.7 of [2] to  $1 - \chi_j$ , we find that  $\sigma_{2^j} \chi_j(x) \geq 1/4$  for any  $x \in \mathbf{I}_j$ . This gives the second assertion of the lemma.  $\square$

We now collapse these polynomials to get as many saturating functions as necessary, with disjoint spectra.

**Lemma 3.2.** *Let  $s \geq 1$ . There exist functions  $g_1, \dots, g_s$  in  $L^p(\mathbb{T})$  and sequences of integers  $(n_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}$ ,  $(m_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}$  satisfying*

- $1 \leq m_{j,r} < n_{j,r} \leq C2^j$  for any  $j$  and any  $r$ ;
- for any  $j \geq j_\alpha$ , any  $x \in \mathbf{I}_j$ , any  $r \in \{1, \dots, s\}$ ,

$$|S_{n_{j,r}} g_r(x) - S_{m_{j,r}} g_r(x)| \geq \frac{C}{j^2} 2^{(j-J)/p}$$

- for any  $r \in \{1, \dots, s\}$ , the spectrum of  $g_r$  is included in  $\bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}] =: G_r$
- if  $r_1 \neq r_2$ ,  $G_{r_1} \cap G_{r_2} = \emptyset$ .

*Proof.* For  $r \in \{1, \dots, s\}$ , we set

$$g_r := \sum_{j \geq j_\alpha} \frac{1}{j^2} e_{(s+r)2^{j+1}} P_j.$$

Define

$$\begin{aligned} m_{j,r} &:= (s+r)2^{j+1} \\ n_{j,r} &:= (s+r)2^{j+1} + (2^{j+1} - 1) \end{aligned}$$

so that each  $g_r$  belongs to  $L^p$  with spectrum included in  $\bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}]$ . Moreover, the intervals  $(m_{j,r}, n_{j,r}]$  are disjoint, so that

$$|S_{n_{j,r}} g_r - S_{m_{j,r}} g_r| = \frac{1}{j^2} |P_j|.$$

Let us also remark that, for any  $j \geq j_\alpha$  and any  $r < s$ ,  $n_{j,r} < m_{j,r+1}$  and  $n_{j,s} < m_{j+1,1}$  so that the spectra  $G_1, \dots, G_s$  are disjoint. This ends up the proof.  $\square$

It is easy to show that, if  $x \in \limsup_j \mathbf{I}_j$ ,  $r \in \{1, \dots, s\}$  and  $\beta < \frac{1}{p} (1 - \frac{1}{\alpha})$ , then

$$\limsup_{n \rightarrow +\infty} \frac{|S_n g_r(x)|}{n^\beta} = +\infty.$$

In some sense, the functions  $g_r$  have the worst possible behaviour on  $\mathbf{I}_j$  if we keep in mind that they have to belong to  $L^p(\mathbb{T})$ . We now show that this property remains true almost everywhere (in the sense of the lebesgue measure) on any affine subspace  $f + \text{span}(g_1, \dots, g_s)$  provided  $s$  is large enough. This is the main step towards the proof of Theorem 1.6.

**3.3. Prevalence of divergence for a fixed divergence index.** We keep the notations of the previous subsection.

**Proposition 3.3.** *Let  $0 < \beta < \frac{1}{p}(1 - \frac{1}{\alpha})$ . There exists  $s \geq 1$  such that, for every  $f \in L^p(\mathbb{T})$ , for almost every  $c = (c_1, \dots, c_s)$  in  $\mathbb{R}^s$ , the function  $g = f + c_1 g_1 + \dots + c_s g_s$  satisfies for every  $x \in D_\alpha$*

$$\limsup_{n \rightarrow +\infty} \frac{|S_n g(x)|}{n^\beta} = +\infty.$$

*Proof.* We set  $\varepsilon = \frac{1}{p}(1 - \frac{1}{\alpha}) - \beta$ . Let  $s > 4/\varepsilon$  and let  $f$  be an arbitrary function in  $L^p(\mathbb{T})$ . For such a value of  $s$ , we will prove the conclusion of the proposition for every  $x \in \limsup_j \mathbf{I}_j$  (recall that  $D_\alpha \subset \limsup_j \mathbf{I}_j$ ).

Let  $M > 0$  and let us introduce

$$S_M := \left\{ g \in L^p(\mathbb{T}); \exists x \in \limsup_{j \rightarrow +\infty} \mathbf{I}_j \text{ s.t. } \forall n \geq 1, |S_n g(x)| \leq M n^\beta \right\}.$$

It is enough to show that for every  $R > 0$ , the set of  $c \in \mathbb{R}^s$  satisfying  $\|c\|_\infty \leq R$  and such that  $f + c_1 g_1 + \dots + c_s g_s$  belongs to  $S_M$  has Lebesgue measure 0. In the sequel, we will fix such values of  $M$  and  $R$ .

If  $j \geq 1$ , we split each interval  $I_{K,j}$  into  $2^j$  subintervals. Each of them has size  $2^{-2j+1}$ , and we get  $2^{J+j}$  intervals  $O_{l,j}$  with  $\bigcup_{l=1}^{2^{J+j}} O_{l,j} = \mathbf{I}_j$ . For  $j \geq 1$ ,  $l \in \{1, \dots, 2^{J+j}\}$ , we set

$$S_M^{(l,j)} := \left\{ g \in L^p(\mathbb{T}); \exists x \in O_{l,j} \text{ s.t. } \forall n \geq 1, |S_n g(x)| \leq M n^\beta \right\}.$$

Clearly,

$$S_M \subset \limsup_{j \rightarrow +\infty} \bigcup_{l=1}^{2^{J+j}} S_M^{(l,j)}$$

and we shall first control the size of the  $c \in \mathbb{R}^s$  with  $\|c\|_\infty \leq R$  such that

$$f + c_1 g_1 + \dots + c_s g_s \in S_M^{(l,j)}.$$

We denote by  $\lambda_s$  the Lebesgue measure on  $\mathbb{R}^s$  and we fix  $j \geq j_\alpha$ ,  $l$  in  $\{1, \dots, 2^{J+j}\}$  and  $c, c^0$  in  $\mathbb{R}^s$  such that  $\|c\|_\infty \leq R$ ,  $\|c^0\|_\infty \leq R$  and

$$\begin{cases} f + c_1 g_1 + \dots + c_s g_s & \in S_M^{(l,j)} \\ f + c_1^0 g_1 + \dots + c_s^0 g_s & \in S_M^{(l,j)}. \end{cases}$$

Let  $r \in \{1, \dots, s\}$  and let us apply the definition of  $S_M^{(l,j)}$  with  $n = n_{j,r}$  and  $n = m_{j,r}$ . The spectra  $(G_l)_{l \neq r}$  being disjoint from  $G_r$ , we can find  $x \in O_{l,j}$  such that

$$|S_{n_{j,r}} f(x) - S_{m_{j,r}} f(x) + c_r (S_{n_{j,r}} g_r(x) - S_{m_{j,r}} g_r(x))| \leq M n_{j,r}^\beta + M m_{j,r}^\beta \leq 2CM 2^{\beta j}.$$

In the same way, we can find  $y \in O_{l,j}$  such that

$$|S_{n_{j,r}} f(y) - S_{m_{j,r}} f(y) + c_r^0 (S_{n_{j,r}} g_r(y) - S_{m_{j,r}} g_r(y))| \leq 2CM 2^{\beta j}.$$

Using the triangle inequality, we get

$$(3) \quad \begin{aligned} & |c_r(S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x)) - c_r^0(S_{n_{j,r}}g_r(y) - S_{m_{j,r}}g_r(y))| \leq \\ & 4CM2^{\beta j} + |S_{n_{j,r}}f(x) - S_{n_{j,r}}f(y)| + |S_{m_{j,r}}f(x) - S_{m_{j,r}}f(y)|. \end{aligned}$$

Now, by combining the norm of the Riesz projection, Nikolsky's inequality and Bernstein's inequality, we know that

$$\|(S_nf)'\|_\infty \leq C(\log n)n^{1+1/p}\|f\|_p$$

(the factor  $\log n$  disappears when  $p > 1$ ). This yields

$$\begin{aligned} |S_{n_{j,r}}f(x) - S_{n_{j,r}}f(y)| & \leq C \log(n_{j,r})n_{j,r}^{1+1/p}|x-y|\|f\|_p \\ & \leq Cj2^{j(1+1/p)}2^{-2j+1}\|f\|_p \\ & \ll 2^{\beta j}. \end{aligned}$$

The same is true for  $|S_{m_{j,r}}f(x) - S_{m_{j,r}}f(y)|$  and we get

$$(4) \quad |c_r(S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x)) - c_r^0(S_{n_{j,r}}g_r(y) - S_{m_{j,r}}g_r(y))| \leq \kappa 2^{\beta j}$$

for some constant  $\kappa$  depending on  $M$  and  $\|f\|_p$  but not on  $j$ .

In the same way,

$$\|(S_ng_r)'\|_\infty \leq C(\log n)n^{1+1/p}\|g_r\|_p \leq C(\log n)n^{1+1/p}.$$

It follows that

$$\begin{aligned} |c_r^0((S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x)) - (S_{n_{j,r}}g_r(y) - S_{m_{j,r}}g_r(y)))| & \leq CRj2^{j(1+1/p)}2^{-2j+1} \\ & \ll 2^{\beta j}. \end{aligned}$$

Combining with (4) we obtain a new constant  $\kappa$  depending on  $M$ ,  $\|f\|_p$  and  $R$  but not on  $j$  such that

$$(5) \quad |(c_r - c_r^0)(S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x))| \leq \kappa 2^{\beta j}.$$

Dividing (5) by  $|S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x)|$  (which is not equal to zero), we find

$$\begin{aligned} |c_r - c_r^0| & \leq \kappa 2^{\beta j} |S_{n_{j,r}}g_r(x) - S_{m_{j,r}}g_r(x)|^{-1} \\ & \leq \frac{\kappa}{C} 2^{\beta j} j^2 2^{-(j-J)/p} \\ & \leq \frac{\kappa 2^{1/p}}{C} j^2 2^{-\varepsilon j} \\ & \leq 2^{-\varepsilon j/2} \end{aligned}$$

provided  $j$  is large enough. Thus, the set of  $c \in \mathbb{R}^s$  with  $\|c\|_\infty \leq R$  and such that  $f + c_1g_1 + \dots + c_sg_s \in S_M^{(l,j)}$  is contained in a ball (for the  $l^\infty$ -norm) of radius  $2^{-\varepsilon j/2}$ . Taking the  $s$ -dimensional Lebesgue measure, this yields

$$\lambda_s \left( \left\{ c \in \mathbb{R}^s; \|c\|_\infty \leq R \text{ and } f + c_1g_1 + \dots + c_sg_s \in S_M^{(l,j)} \right\} \right) \leq 2^s 2^{-\varepsilon sj/2}.$$

This in turn gives

$$\lambda_s \left( \left\{ c \in \mathbb{R}^s; \|c\|_\infty \leq R \text{ and } f + c_1 g_1 + \cdots + c_s g_s \in \bigcup_{l=1}^{2^{J+j}} S_M^{(l,j)} \right\} \right) \leq 2^s 2^{2j - \varepsilon s j / 2}.$$

Thus, since  $\varepsilon s / 2 > 2$ , this last quantity is the general term of a convergent series. Remember that

$$S_M \subset \limsup_{j \rightarrow +\infty} \bigcup_{l=1}^{2^{J+j}} S_M^{(l,j)}.$$

The conclusion of Proposition 3.3 follows from Borel Cantelli's lemma.  $\square$

**Corollary 3.4.** *Let  $\alpha > 1$ . For almost every function  $f$  in  $L^p(\mathbb{T})$ , for every  $x \in D_\alpha$ ,*

$$\limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log n} \geq \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right).$$

*Proof.* This follows immediately from Proposition 3.3, taking a sequence  $(\beta_n)$  increasing to  $\frac{1}{p} \left( 1 - \frac{1}{\alpha} \right)$  and using the fact that a countable intersection of prevalent sets remains prevalent.  $\square$

**3.4. The general case.** We are now able to complete the proof of Theorem 1.6, that is to prove that almost every function  $f \in L^p(\mathbb{T})$  in the sense of prevalence has a multifractal behaviour with respect to the summation of its Fourier series. Indeed, let  $(\alpha_k)_{k \geq 0}$  be a dense sequence in  $(1, +\infty)$ . By Corollary 3.4, for almost every function  $f \in L^p(\mathbb{T})$ , for every  $k \in \mathbb{N}$  and every  $x \in D_{\alpha_k}$ ,

$$\limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log n} \geq \frac{1}{p} \left( 1 - \frac{1}{\alpha_k} \right).$$

Now, let  $\alpha > 1$  and consider a subsequence  $(\alpha_{\phi(k)})_{k \geq 0}$  which increases to  $\alpha$ . Then  $D_\alpha \subset \bigcap_{k \geq 0} D_{\alpha_{\phi(k)}}$  and for any  $x \in D_\alpha$ ,

$$\limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log n} \geq \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right).$$

The conclusion follows now exactly the argument of [2]. For the sake of completeness, we give a complete account. Define

$$\begin{aligned} D_\alpha^1 &= \left\{ x \in D_\alpha; \limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log n} = \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right) \right\} \\ D_\alpha^2 &= \left\{ x \in D_\alpha; \limsup_{n \rightarrow +\infty} \frac{\log |S_n f(x)|}{\log n} > \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right) \right\}, \end{aligned}$$

so that  $\mathcal{H}^{1/\alpha}(D_\alpha^1 \cup D_\alpha^2) = \mathcal{H}^{1/\alpha}(D_\alpha) = +\infty$ . It suffices to prove that  $\mathcal{H}^{1/\alpha}(D_\alpha^2) = 0$ . Let  $(\beta_n)_{n \geq 0}$  be a sequence of real numbers such that

$$\beta_n > \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \beta_n = \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right).$$

Let us observe that

$$D_\alpha^2 \subset \bigcup_{n \geq 0} \mathcal{E}(\beta_n, f).$$

Moreover, Theorem 1.1 for  $p > 1$  and Corollary 2.6 for  $p = 1$  imply that  $\mathcal{H}^{1/\alpha}(\mathcal{E}(\beta_n, f)) = 0$  for all  $n$ . Hence,  $\mathcal{H}^{1/\alpha}(D_\alpha^2) = 0$  and  $\mathcal{H}^{1/\alpha}(D_\alpha^1) = +\infty$ , which proves that

$$\dim_{\mathcal{H}} \left( E \left( \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right), f \right) \right) \geq \frac{1}{\alpha}.$$

By Theorem 1.1 and Corollary 2.6 again, this inequality is necessarily an equality. Finally, such a function  $f$  satisfies the conclusion of Theorem 1.6, setting  $1 - \beta p = 1/\alpha$ .

#### 4. RAPID DIVERGENCE ON BIG SETS FOR FOURIER SERIES OF CONTINUOUS FUNCTIONS

This section is devoted to the proof of Theorem 1.7 and Theorem 1.8. We need to construct functions in  $\mathcal{C}(\mathbb{T})$  for which the Fourier series behave badly on a set with Hausdorff dimension 1. We will construct these functions by blocks. For  $k \geq 1$  and  $\omega > 1$ , we set

$$J_k^\omega := \bigcup_{j=0}^{k-1} \left[ \frac{j}{k} - \frac{1}{2\omega k}, \frac{j}{k} + \frac{1}{2\omega k} \right]$$

which will be seen as a subset of  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . The construction makes use of holomorphic functions, so that we will also see  $\mathbb{T}$  as the boundary of the unit disk  $\mathbb{D}$  and  $J_k^\omega$  as a part of  $\partial\mathbb{D}$ .

**Lemma 4.1.** *There exist three absolute constants  $C_1, C_2, C_3 > 0$  such that, for any  $k \geq 3$ , for any  $\omega \geq \log k$ , one can find a function  $f$  which is holomorphic in a neighbourhood of  $\mathbb{D}$  and which satisfies :*

$$\begin{aligned} (6) \quad & \forall z \in \overline{\mathbb{D}}, \quad \Re f(z) \geq \frac{C_1}{\omega k} \\ (7) \quad & \forall z \in J_k^\omega, \quad |f(z)| \geq C_2 \omega \\ (8) \quad & \forall z \in \mathbb{T}, \quad |f(z)| \leq C_3 \omega \\ (9) \quad & \forall z \in \mathbb{T}, \quad \left| \frac{f'(z)}{f(z)} \right| \leq \omega k. \end{aligned}$$

*Proof.* We set:

$$\begin{aligned} \varepsilon &= \frac{1}{\omega k} \\ z_j &= e^{\frac{2\pi i j}{k}}, \quad j = 0, \dots, k-1 \\ f(z) &= \frac{1}{k} \sum_{j=0}^{k-1} \frac{1 + \varepsilon}{1 + \varepsilon - \overline{z_j} z} \end{aligned}$$

and we claim that  $f$  is the function we are looking for. Indeed, for any  $z \in \overline{\mathbb{D}}$  and any  $j \in \{0, \dots, k-1\}$ ,

$$(10) \quad \Re \left( \frac{1 + \varepsilon}{1 + \varepsilon - \overline{z_j} z} \right) = \frac{1 + \varepsilon}{|1 + \varepsilon - \overline{z_j} z|^2} \Re(1 + \varepsilon - z_j \overline{z}) \geq \frac{1 + \varepsilon}{(2 + \varepsilon)^2} \times \varepsilon \geq C_1 \varepsilon,$$

which proves (6). To prove (7), we may assume that  $z = e^{2\pi i \theta}$  with  $\theta \in [\frac{-\varepsilon}{2}; \frac{\varepsilon}{2}]$ . Then

$$\Re \left( \frac{1 + \varepsilon}{1 + \varepsilon - \overline{z_0} z} \right) = \frac{1 + \varepsilon}{|1 + \varepsilon - z|^2} \Re(1 + \varepsilon - z) \geq \frac{C_2}{\varepsilon}.$$

Moreover, (10) says that for any  $j$ ,  $\Re e \left( \frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z} \right) \geq 0$ . It follows that

$$\Re e f(z) \geq \frac{C_2}{k\varepsilon} = C_2\omega.$$

Conversely, we want to control  $\sup_{z \in \mathbb{T}} |f(z)|$ . Pick any  $z = e^{2\pi i\theta} \in \mathbb{T}$ . By symmetry, we may and shall assume that  $|\theta| \leq \frac{1}{2k}$ . Then we get

$$\left| \frac{1+\varepsilon}{1+\varepsilon-\overline{z_0}z} \right| \leq \frac{C}{\varepsilon}$$

for some constant  $C > 0$ . Now, for any  $j \in \{1, \dots, k/4\}$ , we can write

$$\begin{aligned} |1+\varepsilon-\overline{z_j}z| &\geq |\Im m(\overline{z_j}z)| \\ &\geq \sin \left( \frac{2\pi j}{k} - 2\pi\theta \right) \\ &\geq \frac{2}{\pi} \times 2\pi \left( \frac{j}{k} - \theta \right) \\ &\geq \frac{4}{k} \left( j - \frac{1}{2} \right). \end{aligned}$$

Taking the sum,

$$\left| \sum_{j=1}^{k/4} \frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z} \right| \leq \frac{k(1+\varepsilon)}{4} \sum_{j=1}^{k/4} \frac{1}{j-1/2} \leq Ck \log k$$

(the constant  $C$  may change from line to line). In the same way, we have

$$\left| \sum_{j=3k/4}^{k-1} \frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z} \right| \leq Ck \log k.$$

If  $j \in [k/4, 3k/4]$ , we also have  $|1+\varepsilon-\overline{z_j}z| \geq C$ , so that

$$\left| \sum_{j=k/4}^{3k/4} \frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z} \right| \leq Ck.$$

Putting this together, we get

$$|f(z)| = \left| \frac{1}{k} \sum_{j=0}^{k-1} \frac{1+\varepsilon}{1+\varepsilon-\overline{z_j}z} \right| \leq C \left( \frac{1}{k\varepsilon} + \log k + 1 \right) \leq C_3\omega$$

(this is the place where we need that  $\omega \geq \log k$ ). Finally, it remains to prove (9). We observe that

$$\frac{f'(z)}{f(z)} = \frac{\sum_{j=0}^{k-1} \frac{\overline{z_j}}{(1+\varepsilon-\overline{z_j}z)^2}}{\sum_{j=0}^{k-1} \frac{1}{1+\varepsilon-\overline{z_j}z}}.$$

We deduce that

$$\begin{aligned}
\left| \frac{f'(z)}{f(z)} \right| &\leq \frac{\sum_{j=0}^{k-1} \frac{1}{|1+\varepsilon-\bar{z}_j z|^2}}{\sum_{j=0}^{k-1} \frac{\Re(1+\varepsilon-z_j \bar{z})}{|1+\varepsilon-\bar{z}_j z|^2}} \\
&\leq \frac{\sum_{j=0}^{k-1} \frac{1}{|1+\varepsilon-\bar{z}_j z|^2}}{\sum_{j=0}^{k-1} \frac{\varepsilon}{|1+\varepsilon-\bar{z}_j z|^2}} \\
&\leq \frac{1}{\varepsilon} = \omega k.
\end{aligned}$$

□

The crucial step is given by the following lemma.

**Lemma 4.2.** *Let  $(\varepsilon_n)_{n \geq 1}$  be a sequence of positive real numbers decreasing to zero. Then, if  $n$  is large enough, one can find an integer  $k_n$ , a real number  $\omega_n > 1$  and a trigonometric polynomial  $P_n$  with spectrum in  $[1, 2n-1]$  such that*

- $\|P_n\|_\infty \leq 1$ ;
- For any  $x \in J_{k_n}^{\omega_n}$ ,  $|S_n P_n(x)| \geq \varepsilon_n \log(n)$ .

Moreover, we can choose  $k_n$  and  $\omega_n$  such that  $(k_n)$  goes to  $+\infty$  and  $\omega_n = o(k_n^\alpha)$  for any  $\alpha > 0$ .

*Proof.* It is clear that the conclusion of the lemma is more difficult to obtain when the sequence  $(\varepsilon_n)$  is large. Thus, we may assume that

$$\varepsilon_n \geq \frac{\log \log n}{4\pi \log n}.$$

In particular,  $\varepsilon_n \log n$  goes to infinity. We define  $k_n$  and  $\omega_n$  by

- $\omega_n$  is equal to  $\exp(4\pi(\log n)\varepsilon_n)$
- $k_n$  is the biggest integer  $k$  satisfying

$$2\pi k \omega_n \leq n.$$

Observe that  $\omega_n \geq \log n$  and  $\omega_n = o(n^\alpha)$  for all  $\alpha > 0$ . Then, the inequalities

$$2\pi k_n \omega_n \leq n \leq 2\pi(k_n + 1)\omega_n$$

ensure that

$$k_n \leq n \leq C k_n n^{1/2}$$

if  $n$  is large enough. It follows that  $(k_n)$  goes to  $+\infty$ , that  $\omega_n \geq \log k_n$  and that  $\omega_n = o(k_n^\alpha)$  for any  $\alpha > 0$ .

Let  $f_n$  be the holomorphic function given by Lemma 4.1 for the values  $k = k_n$  and  $\omega = \omega_n$ . We take  $h_n(z) = \log(f_n(z))$ , which defines a holomorphic function in a neighbourhood of  $\overline{\mathbb{D}}$  (remember (6)). Moreover,  $|\Im(h_n(z))| \leq \pi/2$  for any  $z \in \overline{\mathbb{D}}$  and  $h_n(0) = 0$ . Now, we look at the function  $h_n$  on the boundary of the unit disk  $\mathbb{D}$ , that is we introduce the



function  $g_n(x) = h_n(e^{2i\pi x})$  defined on the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . The properties satisfied by  $f_n$  translate into

$$\begin{aligned} \forall x \in J_{k_n}^{\omega_n}, \quad |g_n(x)| &\geq \log \omega_n + \log C_2 \\ \forall x \in \mathbb{T}, \quad |g_n(x)| &\leq \log \omega_n + \log C_3 \\ \forall x \in \mathbb{T}, \quad |g'_n(x)| &\leq 2\pi k_n \omega_n \leq n. \end{aligned}$$

We apply Lemma 1.7 of [2], which is a precised version of Féjer's theorem, to the function  $\theta_x(t) = g_n(t) - g_n(x)$  for  $x \in \mathbb{T}$ . Since  $\|\theta_x\|_\infty \leq 2\log \omega_n + 2\log C_3$ ,  $\|\theta'_x\|_\infty \leq n$  and  $\theta_x(x) = 0$ , we get

$$|\sigma_n \theta_x(x)| \leq \frac{1}{2} \log \omega_n + C_4$$

for some absolute constant  $C_4$ . If  $x \in J_{k_n}^{\omega_n}$  we deduce that

$$|\sigma_n g_n(x)| \geq \frac{1}{2} \log \omega_n - C_5.$$

Finally we set

$$P_n = \frac{2}{\pi} e_n \sigma_n (\Im m g_n) = \frac{2}{\pi} e_n \Im m (\sigma_n g_n),$$

so that  $\|P_n\|_\infty \leq 1$ . Now, remember that  $g_n$  is the restriction to the circle of an holomorphic function  $h_n$  satisfying  $h_n(0) = 0$ . We can then write  $\sigma_n g_n = \sum_{j=1}^{n-1} a_j e_j$ , so that  $2i \Im m \sigma_n g_n = -\sum_{j=1}^{n-1} \overline{a_j} e_{-j} + \sum_{j=1}^{n-1} a_j e_j$ . Thus, the spectrum of  $P_n$  is contained in  $[1, 2n-1]$ . Moreover, for any  $x \in J_{k_n}^{\omega_n}$ , we get

$$\begin{aligned} |S_n P_n(x)| &= \frac{1}{\pi} \left| \sum_{j=1}^{n-1} \overline{a_j} e_{-j+n} \right| \\ &= \frac{1}{\pi} |\sigma_n g_n(x)| \\ &\geq \frac{1}{2\pi} \log \omega_n - C_6 \\ &= 2\varepsilon_n \log n - C_6 \\ &\geq \varepsilon_n \log n \end{aligned}$$

if  $n$  is large enough. □

We are now ready to construct the dense  $G_\delta$ -set of functions required in Theorem 1.7.

*Proof of Theorem 1.7.* Let  $(\delta_n)_{n \geq 2}$  be a sequence going to 0. We first consider an auxiliary sequence  $(\delta'_n)_{n \geq 1}$  such that

$$\lim_{n \rightarrow +\infty} \delta'_n = 0, \quad \lim_{n \rightarrow +\infty} \frac{\delta'_n}{\delta_n} = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \delta'_n \log n = +\infty.$$

Let  $(g_n)_{n \geq 1}$  be a dense sequence in  $\mathcal{C}(\mathbb{T})$ , such that the spectrum of  $g_n$  is contained in  $[-n, n]$ . We set  $\eta_n = \max(\delta'_k; n \leq k)$ . The sequence  $(\eta_n)_{n \geq 1}$  decreases to zero. Moreover, we fix a sequence  $(\varepsilon_n)_{n \geq 1}$ , going to zero, such that  $\varepsilon_n/\eta_n$  tends to infinity. Lemma 4.2 gives us an integer  $N$ , a sequence  $(P_j)_{j \geq N}$  of trigonometric polynomials with spectrum

contained in  $[1, 2j-1]$ , a sequence  $(k_j)_{j \geq N}$  of integers going to  $+\infty$  and a sequence  $(\omega_j)_{j \geq N}$  satisfying  $\omega_j > 1$ , such that

$$|S_j P_j(x)| \geq \varepsilon_j \log j$$

for any  $x \in J_{k_j}^{\omega_j}$ . Moreover, we can choose  $\omega_j$  such that  $\omega_j = o(k_j^\alpha)$  for any  $\alpha > 0$ . Let us define for  $j \geq N$

$$h_j := g_j + \frac{\eta_j}{\varepsilon_j} e_j P_j.$$

The sequence  $(h_j)_{j \geq N}$  remains dense in  $\mathcal{C}(\mathbb{T})$ . Let us also observe that the spectra of  $g_j$  and  $\frac{\eta_j}{\varepsilon_j} e_j P_j$  are disjoint. It follows that if  $x \in J_{k_j}^{\omega_j}$ ,

$$|S_{2j} h_j(x) - S_j h_j(x)| = \left| \frac{\eta_j}{\varepsilon_j} S_j P_j(x) \right| \geq \eta_j \log j.$$

Thus, for any  $x \in J_{k_j}^{\omega_j}$ , one may find  $n \in \{j, 2j\}$  such that

$$|S_n h_j(x)| \geq \frac{1}{2} \eta_j \log j \geq \frac{1}{2} \delta'_n (\log n - \log 2).$$

Let  $r_j > 0$  be small enough so that

$$|S_n h(x)| \geq |S_n h_j(x)| - 1$$

for any  $h \in B(h_j, r_j)$  and any  $n \in \{j, 2j\}$  (the open balls are related to the norm  $\|\cdot\|_\infty$ ). Then, we claim that the following dense  $G_\delta$ -set of  $\mathcal{C}(\mathbb{T})$  fulfills all the requirements:

$$G := \bigcap_{p \geq N} \bigcup_{j \geq p} B(h_j, r_j).$$

Indeed, pick any  $h$  in  $G$  and any increasing sequence  $(j_p)$  such that  $h$  belongs to  $B(h_{j_p}, r_{j_p})$ . Setting  $\rho_p = \omega_{j_p}$  and  $s_p = k_{j_p}$ , it is not hard to show that

$$E := \limsup_{p \rightarrow +\infty} E_p, \text{ with } E_p = J_{s_p}^{\rho_p}$$

has Hausdorff dimension 1. Indeed, remember that for any  $\alpha > 0$ ,  $\omega_j = o(k_j^\alpha)$ . It follows for any  $\alpha > 0$  and for  $p$  large enough,  $E_p$  contains

$$F_p = \bigcup_{j=0}^{s_p-1} \left[ \frac{j}{s_p} - \frac{1}{2s_p^{1+\alpha}}; \frac{j}{s_p} + \frac{1}{2s_p^{1+\alpha}} \right],$$

Now, it is well-known that  $\limsup_p F_p$  has Hausdorff dimension equal to  $1/(1+\alpha)$  (this follows for instance from the mass transference principle of [3]). Finally,  $\dim_{\mathcal{H}}(E) \geq \frac{1}{1+\alpha}$ . Moreover, for any  $x \in E$ , the work done before and the fact that  $\delta'_n \log n$  goes to  $+\infty$  show that

$$|S_n h(x)| \geq \frac{1}{2} \delta'_n (\log n - \log 2) - 1 \geq \frac{1}{4} \delta'_n \log n$$

for infinitely many values of  $n$ . We then get

$$\frac{|S_n h(x)|}{\delta_n \log n} \geq \frac{\delta'_n}{4\delta_n}$$

for infinitely many values of  $n$ . This achieves the proof of Theorem 1.7.  $\square$

We can finally construct the prevalent set of functions required in Theorem 1.8.

*Proof of Theorem 1.8.* Let  $(\delta_n)_{n \geq 2}$  be a sequence going to 0 and denote by  $A$  the set of continuous functions  $f \in \mathcal{C}(\mathbb{T})$  such that

$$\dim_{\mathcal{H}} \left( \left\{ x \in \mathbb{T} ; \limsup_{n \rightarrow +\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty \right\} \right) < 1.$$

We have to prove that  $A$  is Haar-null in  $\mathcal{C}(\mathbb{T})$ .

Let  $f_0$  be a fixed function in the complementary of  $A$  (such a function does exist by Theorem 1.7) and let  $g$  be an arbitrary function in  $\mathcal{C}(\mathbb{T})$ . Suppose that  $t_1$  and  $t_2$  are two real numbers such that

$$t_1 f_0 \in (g + A) \quad \text{and} \quad t_2 f_0 \in (g + A).$$

We can then find  $f_1 \in A$  and  $f_2 \in A$  such that  $(t_1 - t_2)f_0 = f_1 - f_2$ . It is clear that  $f_1 - f_2 \in A$  ( $A$  is a vector subspace of  $\mathcal{C}(\mathbb{T})$ ). It follows that  $t_1 = t_2$ , so that

$$\#(\text{span}(f_0) \cap (g + A)) \leq 1.$$

In particular, the Lebesgue-measure in  $\text{span}(f_0)$  is transverse to  $A$  and  $A$  is Haar-null in  $\mathcal{C}(\mathbb{T})$ .  $\square$

*Remark:* We have just only proved that a proper subspace in a complete metric vector space is Haar-null. This property is probably well-known.

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CLERMONT UNIVERSITÉ, UNIVERSITÉ BLAISE PASCAL, LABORATOIRE DE MATHÉMATIQUES, BP 10448, F-63000 CLERMONT-FERRAND - CNRS, UMR 6620, LABORATOIRE DE MATHÉMATIQUES, F-63177 AUBIERE

*E-mail address:* Frederic.Bayart@math.univ-bpclermont.fr, Yanick.Heurteaux@math.univ-bpclermont.fr